2937. [2004 : 174, 176] *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Suppose that $x_1, \ldots, x_n$ $(n \geq 2)$ are positive real numbers. Prove that

$$(x_1^2 + \cdots + x_n^2)\left(\frac{1}{x_1^2 + x_1x_2} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right) \;\geq\; \frac{n^2}{2}\,.$$

I. *Composite of essentially the same solution by Arkady Alt, San Jose, CA, USA; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; John G. Heuver, Grande Prairie, AB; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Since

$$\begin{aligned} &2(x_1^2 + x_2^2 + \cdots + x_n^2) - 2(x_1x_2 + x_2x_3 + \cdots + x_nx_1) \\ &\qquad = \;(x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_n - x_1)^2 \;\geq\; 0\,, \end{aligned}$$

we have $x_1^2 + x_2^2 + \cdots + x_n^2 \geq x_1x_2 + x_2x_3 + \cdots + x_nx_1$. Hence,

$$\begin{aligned} &2(x_1^2 + x_2^2 + \cdots + x_n^2)\left(\frac{1}{x_1^2 + x_1x_2} + \frac{1}{x_2^2 + x_2x_3} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right) \\ &\qquad \geq \;\Big((x_1^2 + x_1x_2) + (x_2^2 + x_2x_3) + \cdots + (x_n^2 + x_nx_1)\Big) \cdot \\ &\qquad\qquad \cdot \left(\frac{1}{x_1^2 + x_1x_2} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right)\,. \end{aligned}$$

The right side is at least n^2, by the AM–HM Inequality.

Clearly, equality holds if and only if all the x_i's are equal.